Back Paper Examination : Topology II. MMath I

Max. Marks : 100 Time : 3 hours

You may use theorems/propositions proved in the class after correctly stating them. Any other claim must be accompanied by a proof.

(1) When do you say a continuous map $p : E \longrightarrow B$ between spaces is a covering map? Decide whether the following maps are covering maps.
 (a) $p : (0, \infty) \longrightarrow S^1$, $p(t) = e^{2\pi i t}$,
 (b) $p : \mathbb{C} \longrightarrow \mathbb{C}$, $p(cos\, t, sin\, t) = (3cos\, t + 2sint\, t, cos\, t + sin\, t)$. [2+4+4]

(2) State the definition of a covering space. Let X, Y be connected, locally path connected and semi-locally simply connected spaces and let $p : \widetilde{X} \longrightarrow X$, $q : \widetilde{Y} \longrightarrow Y$ be the universal covers.
 (a) Show that for any continuous map $f : X \longrightarrow Y$, there exists a continuous map $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ such that $q \circ \widetilde{f} = f \circ p$.
 (b) Is such a map $\widetilde{f}$ unique?
 (c) Assume that f in injective. Show that $\widetilde{f}$ is injective if and only if $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$ is injective for some x. [2+4+4+4]

(3) Sketch all 2-sheeted coverings of $S^1 \vee S^1$ up to covering space isomorphism (without base points). Give complete justifications. [10]

(4) Define the term *deformation retract*. Show that S^n is a deformation retract of $\mathbb{R}^{n+1} - 0$. [2+8]

(5) Compute the homology groups $H_i(S^n; \mathbb{Z})$, $i \geq 0$, $n \geq 0$. [10]

(6) State the definition of a CW-complex.
 (a) Prove that a CW-complex is compact if and only if it is finite.
 (b) Show that $\mathbb{R}P^n$ is a CW-complex with one i-cell for each i, $0 \leq i \leq n$. [26+8]

(7) When do you say that a pair of topological spaces (X, A) is a good pair?
 (a) Show that $(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ is a good pair.
 (b) Prove that if (X, A) is a good pair, then the quotient map $p : X \longrightarrow X/A$ induces an isomorphism
 $$p_* : H_i(X, A; \mathbb{Z}) \longrightarrow \widetilde{H}_i(X/A; \mathbb{Z})$$
 for all $i \geq 0$.
 (c) Show that the quotient map $p : \mathbb{R}P^3 \longrightarrow \mathbb{R}P^3/\mathbb{R}P^2$ is not null homotopic. [2+4+6+6]

(8) Define the degree of a map $f : S^n \longrightarrow S^n$. Show that the degree is independent of the choices made in the definition.
 (a) Let $f : S^2 \longrightarrow S^2$ be a map that has no fixed points. Show that degree of f is -1.
 (b) Let $f : S^n \longrightarrow S^n$ be a map that satisfies $f(x) = f(-x)$ for all x. Show that f has even degree. [2+4+6]